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Statistical properties of Weibull estimators

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The Weibull parameters were estimated for data produced by Monte Carlo simulations using three different approaches: linear regression, moments method, and maximum likelihood method. The last of these was shown to be the most appropriate approach for the whole range of sample sizes of 4 to 100 for estimating the Weibull parameters of a brittle material. In each simulation 10 000 estimators were produced. Using these values histograms of the estimators were created, which showed the asymmetry of the Weibull modulus distribution. The integrals of these density functions were directly used to determine confidence intervals for the estimated Weibull moduli. Furthermore it was reaffirmed that a minimum of 30 samples are required for a good characterization of the strength of a brittle material.

1. Introduction

Weibull statistics has become a well-established characterization tool in the field of fracture strength of ceramics. Based on physical assumptions [1, 2], the Weibull equation describes the relationship between the probability of failure P_f of a perfectly elastic body under a uniaxial tensile stress of σ. It thus predicts the inherent dispersion in fracture strength of brittle materials. The simplified two-parameter Weibull equation

$$P_f \equiv F(\sigma) = 1 - \exp\left[- \left(\frac{\sigma}{\sigma_0}\right)^m \right] \quad (1)$$

has been widely used in estimating chance of failure of ceramic components.

The two parameters in Equation 1, the so called Weibull parameters, determine the shape and location of the cumulative distribution function $F(\sigma)$. The Weibull modulus m, sometimes called the shape parameter, has a value between 5 and 20 for technical ceramics. On a normalized scale, a higher m leads to a steeper function and thus a lower dispersion of fracture stresses. The scale parameter σ_0 is closely related to the mean fracture stress $\bar{\sigma}$ through

$$\bar{\sigma} = \sigma_0 \Gamma\left(1 + \frac{1}{m}\right) \quad (2)$$

with $\Gamma(x)$ taking values between 0.9 and 1 for the above m-interval. Appearing in the denominator of the exponential term – similar to the square of standard deviation in a normal distribution – the parameter σ_0 influences the variance of the fracture stress, i.e. the steepness of the function: a smaller σ_0 means – on an absolute scale – a lower dispersion. Shih [3] demonstrates the influence and mutual interaction of the Weibull parameters on the form of the distribution function with many examples.

Once a set of N experimentally measured fracture stresses are obtained, it is desirable to fit the Weibull equation (Equation 1) to these observations, i.e. to determine the two parameters m and σ_0, knowledge of which leads to complete characterization of the material for the given volume.

2. Estimation of the Weibull parameters

There are different approaches in estimation of the two Weibull parameters. The methods usually employed will first be shortly discussed.

2.1. Method of linear regression

Linear regression is a special case of the least-squares method. Taking the logarithm of Equation 1 twice gives a linear equation:

$$\ln\left[\ln\left(\frac{1}{1 - P_{f,n}}\right)\right] = m \ln\sigma_n - m \ln\sigma_0 \quad (3)$$

with the slope $b = m$ and a y-intercept of $a = -m\ln\sigma_0$. The σ_n values are the experimentally determined fracture stresses ordered as follows:

$$\sigma_1 < \sigma_2 < \ldots < \sigma_n < \ldots < \sigma_{N-1} < \sigma_N$$

A probability of fracture will be assigned to each σ_n such that

$$P_{f,1} < P_{f,2} < \ldots < P_{f,n} < \ldots < P_{f,N-1} < P_{f,N}$$

where $0 < P_f < 1$. Since the true value of $P_{f,n}$ for each σ_n is not known, it has to be estimated. This estimator is to be chosen such that on average, the errors arising each time due to this estimation compensate each other. There is more than one definition for the probability of fracture (see e.g. [4–7]). Four of the most common designations are

$$P_{f,a} = \frac{n - 0.5}{N} \quad (4a)$$

$$P_{f,b} = \frac{n}{N + 1} \quad (4b)$$

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$$P_{f,c} = \frac{n - 0.3}{N + 0.4} \quad (4c)$$

$$P_{f,d} = \frac{n - 0.5}{N + 0.25} \quad (4d)$$

for the nth fracture, from a total of N results. The most common definition is that of $P_{f,b}$ (Equation 4b).

2.2. Method of moments

A set of data (or a distribution) may be reduced to a few numbers through calculating its moments. The first moment results in the mean value, and the standard deviation may be calculated from the second moment of a distribution.

The derivative of the cumulative distribution function (CDF) of the Weibull distribution (Equation 1) gives the probability density function (PDF):

$$f(\sigma) = \frac{dP_f}{d\sigma} = \frac{dF(\sigma)}{d\sigma} = \frac{m}{\sigma_0}\left(\frac{\sigma}{\sigma_0}\right)^{m-1} \times \exp\left[-\left(\frac{\sigma}{\sigma_0}\right)^m\right] \quad (5)$$

Putting $f(\sigma)$ of the Weibull distribution in the general definitions of average $\bar{x}$ and variance S_x^2 of a distribution (see e.g. [8]), one can calculate the first two moments of the Weibull distribution:

$$\bar{\sigma} = \sigma_0 \Gamma\left(1 + \frac{1}{m}\right) \quad (2)$$

$$S_\sigma^2 = \sigma_0^2\left\{\Gamma\left(1 + \frac{2}{m}\right) - \left[\Gamma\left(1 + \frac{1}{m}\right)\right]\right\} \quad (6)$$

The standard deviation is the square root of the variance. The coefficient of variation C_{var} for the Weibull distribution is thus

$$C_{var,\sigma} \equiv \frac{S_\sigma}{\bar{\sigma}} = \sigma_0\{\Gamma(1 + \tfrac{2}{m}) - [\Gamma(1 + \tfrac{1}{m})]^2\}^{1/2} / [\sigma_0 \Gamma(1 + \tfrac{1}{m})] \quad (7)$$

In the method of moments it is assumed that the mean and variance of the experimental data equal those of the whole distribution, i.e. infinite number of samples. Setting the mean and variance of the experimental data in Equation 7, the parameter σ_0 drops, and Equation 7 becomes thus a function of m only and can be solved for m using an iterative procedure, e.g. the Newton–Rhapson method [9].

2.3. Maximum likelihood method

In this approach values for the two parameters m and σ_0 are sought which result in a Weibull function which describes the experimental data that are most likely. The probability that for an estimated set of Weibull parameters, the experimental results *would have occurred*, should be maximized. This is equal to the probability that all σ_n values occur simultaneously, i.e. the product of all fracture probabilities:

$$f_N = f(\sigma_1)f(\sigma_2)\ldots f(\sigma_n)\ldots f(\sigma_{N-1})f(\sigma_N)$$

which is defined as the likelihood function L:

$$L \equiv f_N = \prod_n f(\sigma_n) \quad (8)$$

This function should be maximized. To maximize the likelihood function L, the partial derivatives with respect to m and σ_0 are set equal to zero. Since taking the derivative of a sum is easier than that of a product, the derivation is done on the logarithm of L.

The detailed calculation (for both the moments and the maximum likelihood) can be found in the work of many authors, e.g. the appendix in Khalili [10]. Similarly to the case of the moments method, the result is an equation

$$\frac{N}{m} + \sum_{n=1}^{N} \ln \sigma_n - N\frac{\sum_{n=1}^{N} \sigma_n^m \ln \sigma_n}{\sum_{n=1}^{N} \sigma_n^m} = 0 \quad (9)$$

in which only the parameter m and the experimental data $\{\sigma_1, \ldots, \sigma_n, \ldots, \sigma_N\}$ appear. Equation 9 may be solved for m with an iteration method.

3. Monte Carlo simulations as a means of comparing the evaluation methods

It is interesting to find out which method of evaluation results in the most accurate estimation of the Weibull parameters. A quantitative approach has not yet been presented. Instead, Monte Carlo simulations have been used to compare the different methods of estimation. Other authors [11–14] have reported simulation results of studies on selected methods. However, either the simulation process was not reported in full detail, such that a full re-examination could not be followed, or too few repetitions of sampling were done. The aim of this paper is to contribute supplementary information towards a more comprehensive comparison of the methods mentioned above, in order to choose the appropriate estimation method in evaluation of the Weibull function.

3.1. The simulation procedure

Suppose we have a material whose fracture stress variation follows a Weibull distribution of known parameters. That is, the exact values of both parameters m and σ_0 of the "parent population" are known: m_{true} and $\sigma_{0,true}$. If we now choose N random samples of this material, measure the fracture stress of each sample, and then evaluate the Weibull function for this particular experimental set, i.e. determine the estimated parameters m and σ_0, we will definitely not obtain m_{true} and $\sigma_{0,true}$ as results of the estimation procedure. This is simply due to the fact that the true parameters of a distribution are only known when an infinite number of samples are tested.

In Fig. 1 the procedure of Monte Carlo simulations presented in this study is shown. A random generator*

*The subroutine "ran0" from Press *et al.* [9] was used. For every program run (containing N times 10000 values) the random generator starts with the same number. To avoid repetition of the same set of fracture stresses for all data a given number X (typed as input from operator) of dummy variables were produced at the beginning of each program and the actual simulation would then start with the $(X + 1)$th random variable set equal to P_f.

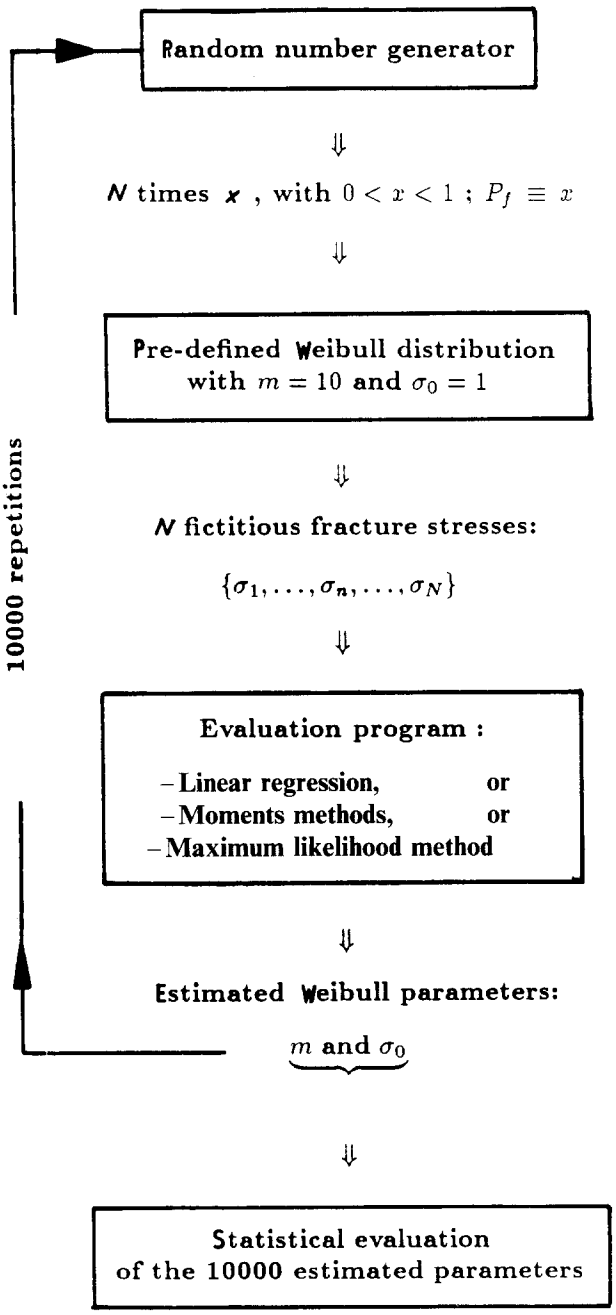


Figure 1 Schematic flow diagram of the Monte Carlo simulations.

produces a real number in the interval [0, 1]. This number will be taken as the probability of fracture $P_{f,n}$ and set in Equation 1, where the known values of m_{true} and $\sigma_{0,true}$ (here $m = 10$ and $\sigma_0 = 1$) are already inserted, and solved for σ_n. If one repeats this procedure N times, there results a set of fictitious fracture stresses, $\{\sigma_1, \ldots, \sigma_n, \ldots, \sigma_N\}$. These may then be treated as experimental results, the probability of fracture and the underlying Weibull function of which are unknown. The Weibull function of this set can be evaluated, using one of the above-mentioned methods. Repeating this procedure many times gives a set of estimators m and σ_0, whose distribution around the true value may be statistically characterized.

In the present work we repeated the above procedure 10 000 times for each method (unless otherwise mentioned) and each sample size N in order to ensure statistical convergence of the results. At the early stage of this study some simulations were done with only 1500 repetitions, the results of which are still representative. To study the effect of the number of specimens, the sample size N was increased progressively from 4 to 100.

4. Simulation results

4.1. The properties of the estimator m

The simplest way of representing a set of data is presumably to take their average. Fig. 2 shows the dependence of the normalized estimated parameter m_{ave}/m_{true} on sample size N for the three evaluation methods (1500 repetitions). For the case of linear regression two different definitions were set for the probability of fracture P_f. Fig. 3 shows the same parameter as a function of inversed sample size (10 000 repetitions). Figs 2 and 3 suggest that

(i) the average value of the estimators approaches the true value m_{true} with increasing sample size N;

(ii) the linear regression estimator m_{LR} with the definition $P_f = (n - 0.5)/N$, the moments estimator m_{Mom}, and the maximum likelihood estimator m_{ML} all converge to the true value, but m_{LR} with the definition $P_f = n/(N + 1)$, which on average underestimates the value of m for all sample sizes, does not converge to the true value;

(iii) the average estimate is extremely sensitive to sample size, particularly for the range $N < 30$;

(iv) the maximum likelihood estimation results on average in the largest overestimation: $m_{ave,ML} > m_{ave,Mom} > m_{ave,LR} > m_{true}$.

The coefficient of variation, a common measure for the breadth of a distribution, may also be defined for the parameter m:

$$C_{var,m} \equiv \frac{\text{Std. deviation}}{\text{Mean}} = \frac{S_m}{m_{ave}} \qquad (10)$$

Since the standard deviation is the dispersion of all elements of a distribution about the mean, it seems particularly appropriate to evaluate the coefficient of variation which represents the breadth with reference to the mean (i.e. as a relative value), instead of the standard deviation. Fig. 4 shows the coefficient of variation C_{var} as a function of the inverse square root of sample size, $1/N^{1/2}$ for four different definitions of probability of fracture in the linear regression estimation (1500 repetitions). Similarly to the average value of the estimators, C_{var} decreases with increasing sample size, i.e. the more samples one measures the more accurate the result.

For larger sample sizes, all four estimators lie very close to the approximation line of slope 1, in agreement with previous authors [13, 14]. Comparison of the behaviour of C_{var} for the four different definitions in Fig. 4 shows that the choice of definition in unimportant. Thus in the following sections only $P_f = (n - 0.5)/N$ will be considered, in agreement with other authors (e.g. [14]).

Fig. 5 depicts the values of C_{var} for the estimators of maximum likelihood and moments method, in addition to the best estimator of linear regression (see

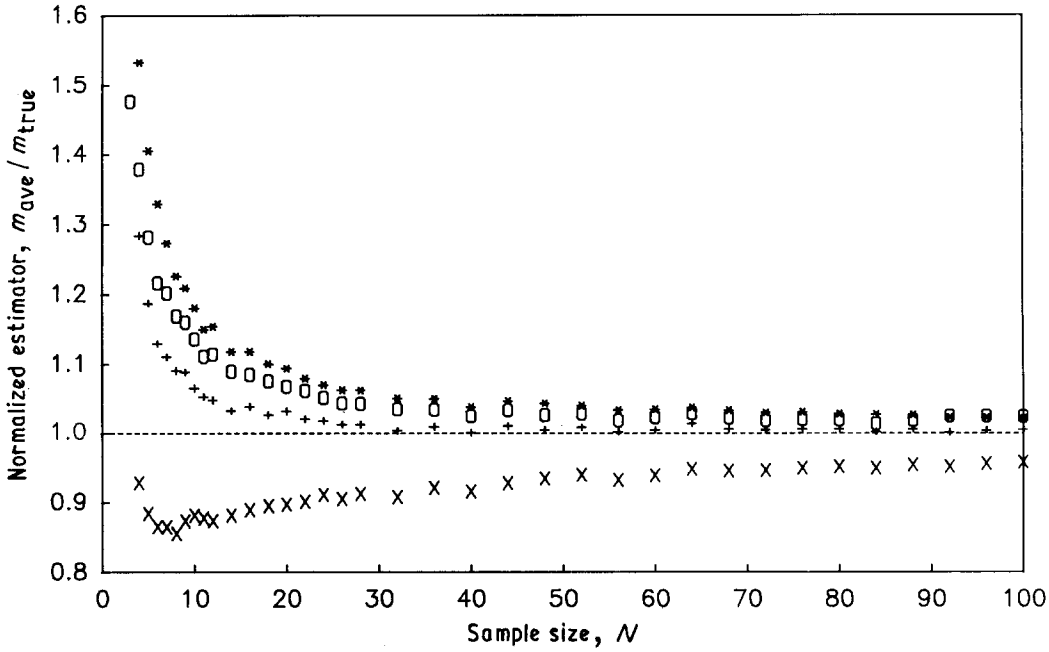


Figure 2 Estimated Weibull modulus as a function of sample size: (×) linear regression, $P_f = n/(N + 1)$; (+) linear regression, $P_f = (n - 0.5)/N$; (□) moments method; (∗) maximum likelihood method.

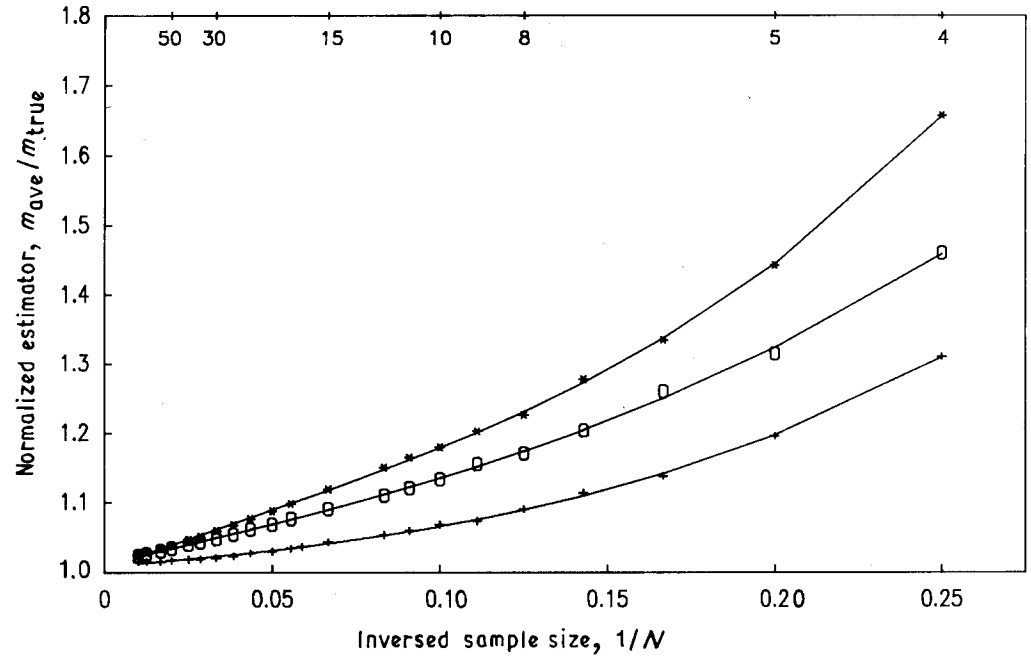


Figure 3 Estimated Weibull modulus versus inverse of sample size: (+) linear regression, $P_f = (n - 0.5)/N$; (□) moments method; (∗) maximum likelihood method.

below). C_{var} is likewise sensitive to sample size, especially for $N < 30$. The fitted polynomial goes through zero for $N = \infty$, i.e. the true value of the parameter m is obtained only for an infinite number of samples. For any smaller amount, only an estimate but not the true value can be achieved.

It is clear from Fig. 5 that the maximum likelihood estimator results in the smallest C_{var} for the whole range of sample size. Furthermore, Fig. 5 shows that in order to have an equal measure of accuracy of an estimate, a smaller number of samples is required if one uses the maximum likelihood evaluation rather than the other methods.

Fig. 2 suggests that a minimum amount of 30 specimens is required to have an acceptable degree of accuracy in obtaining the material parameters. The curvatures of all curves in Figs 4 and 5 decrease continuously with increasing number of specimens. This means that up to about 30 specimens, there is a high gain in accuracy for each additional sample; from about 30 samples upwards, the gain in precision decreases with increasing sample size.

The first two moments of a distribution, i.e. the average and standard deviation, are accurate descriptions of the central tendence and dispersion of the particular distribution: this is especially true for the

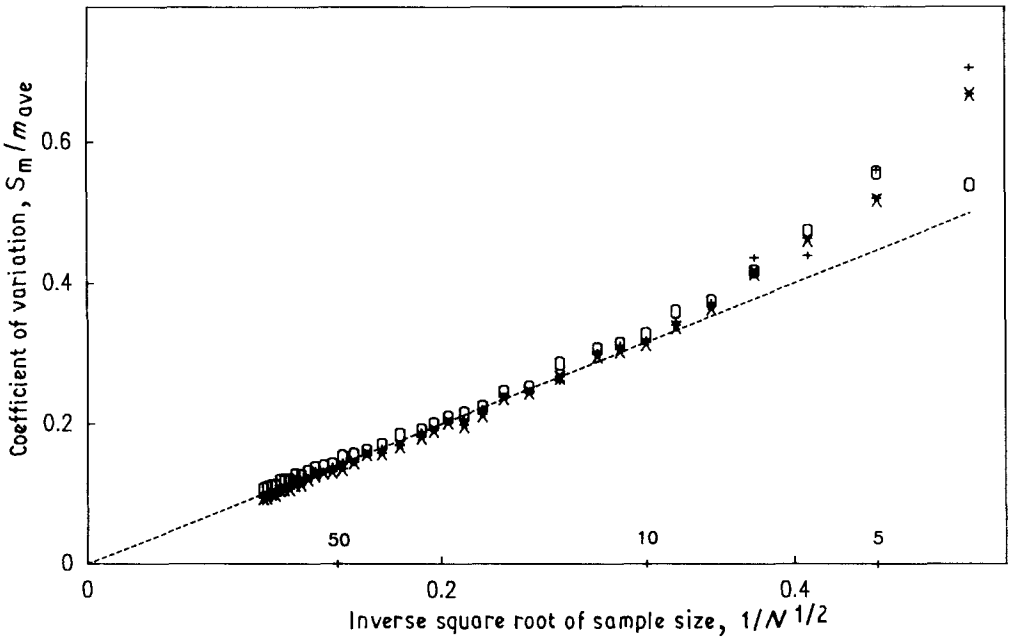


Figure 4 Coefficient of variation of m versus inverse square root of sample size for the method of linear regression (see Equations 4a–d): (+) $P_{f,a}$, (□) $P_{f,b}$, (*) $P_{f,c}$, (×) $P_{f,d}$; (---) approximation $C_{var} = 1/N^{1/2}$.

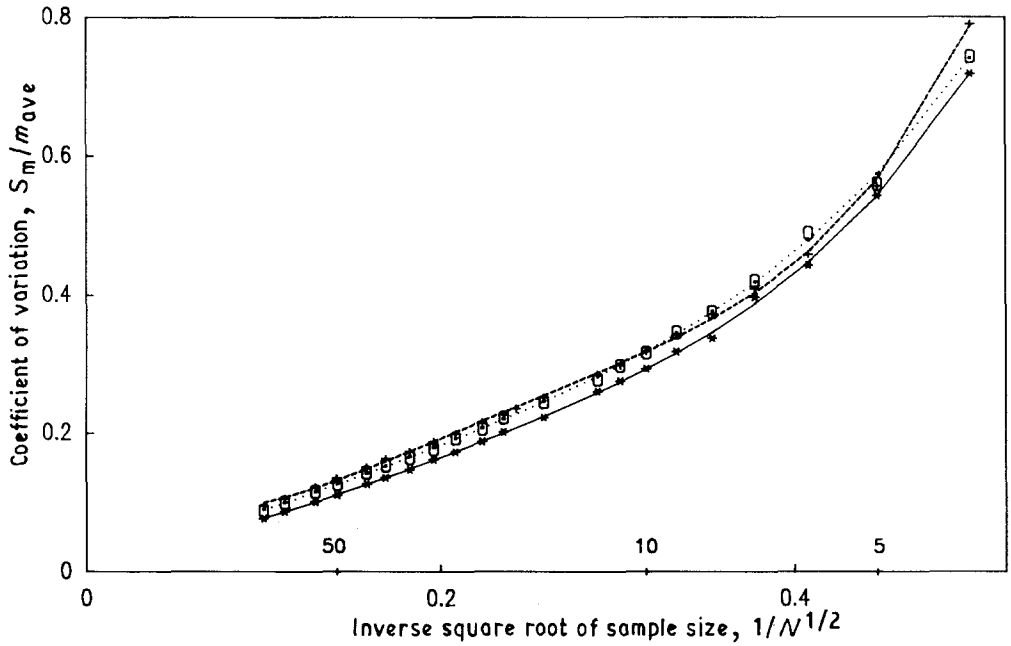


Figure 5 Coefficient of variation of m versus inverse square root of sample size: (+) linear regression, $P_f = (n - 0.5)/N$; (□) moments method; (*) maximum likelihood method.

normal distribution. Previously it has not been shown whether the approximated m values from Monte Carlo simulations also follow a normal distribution; this has been merely assumed. To find out the distribution of the approximated m values, the 10 000 values were stored and then classified into groups falling in a definite interval.

The range between the highest and the lowest estimator ($m_{max} - m_{min}$) was divided into 35 equally sized intervals. The number of estimators falling into each interval was counted. This number, normalized through division by 10 000, the total number of estimators, produces the relative frequency of occurrence, the y-value. The x-value was simply the mid-point of the given interval.

The resulting histograms (Figs 6–11) are thus point diagrams produced by this procedure and can be seen as empirical density functions of the actual estimator. No curve fitting was done; the depicted curves are merely point distributions. This procedure was done only for a constant specimen size of $N = 30$. According to Figs 4 and 5, the results can be expected to be valid for the whole range of sample sizes examined in this work.

Fig. 6 depicts the PDFs for the method of linear regression, with four different definitions of probability of fracture (Equations 4a–d) for a sample size of $N = 30$. The striking feature is the asymmetry of the density function for parameter m. There is a definite lower limit, a minimum value of m, but the parameter

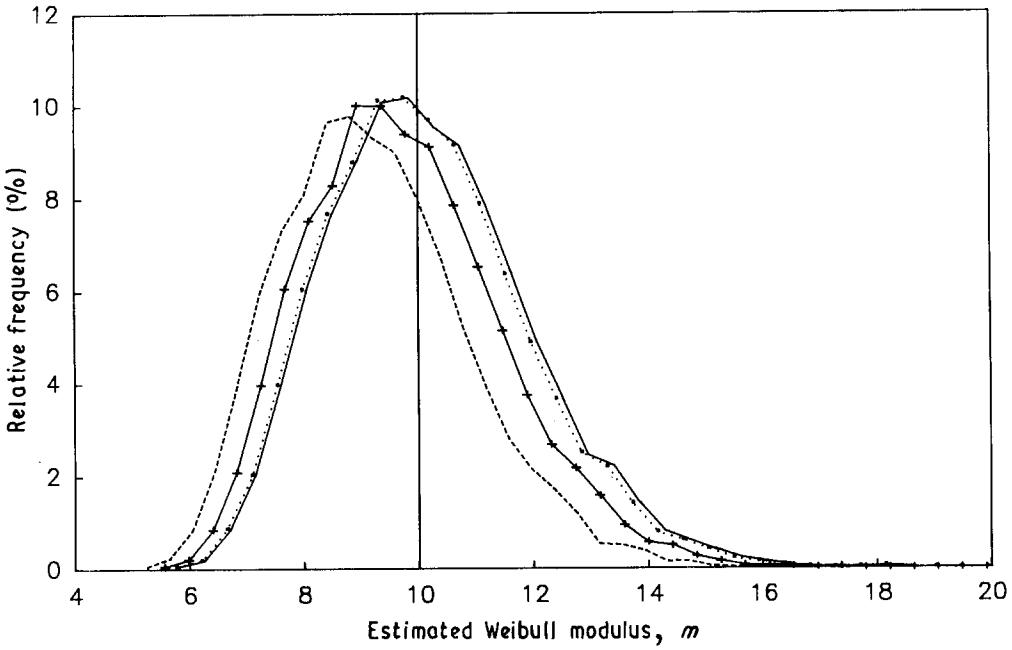


Figure 6 Probability density function of Weibull modulus for the method of linear regression for a constant sample size $N = 30$ (see Equations 4a–d): (——) $P_{f,a}$, (– – –) $P_{f,b}$, (+) $P_{f,c}$, (· · ·) $P_{f,d}$.

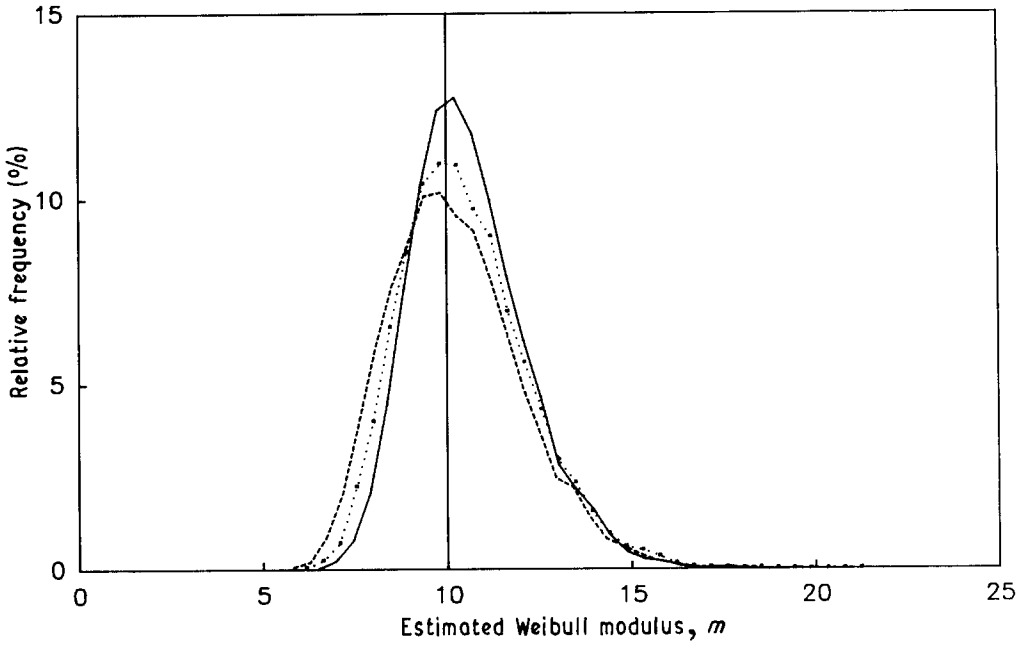


Figure 7 Probability density function of Weibull modulus, $N = 30$: (– – –) linear regression, $P_f = (n - 0.5)/N$; (· · ·) moments method; (——) maximum likelihood method.

is sometimes drastically overestimated. This is to be expected since m may not have any negative values, i.e. there exists a lower bound (zero) for the estimator but no upper limit.

Fig. 6 shows that the most common definition for P_f, Equation 4b, results in the least acceptable outcome: the majority of estimated parameters are fairly distant from the true value $m = 10$. If linear regression is to be chosen as the evaluation method, it should only be with the P_f definition of Equation 4a, or the closely related definition of Equation 4d, since according to both Fig. 6 and Fig. 4 there is no significant difference in their dispersion. Due to the asymmetry of the m distribution, the average is no longer acceptable as a characterizing parameter, since a few outlying larger estimators pull the average to higher values. The median, defined as the value at which there are as many points above as below it, i.e. the $(x/2)$th value of x ordered variables, seems more representative for the data.

In Fig. 7 the best curve from Fig. 6 and the density functions of moments and maximum likelihood estimators are shown for $N = 30$. The distribution curve of the maximum likelihood method is the narrowest of all for a constant sample size of 30. Following Fig. 5, this should hold for the whole range of $N = 4$ up to

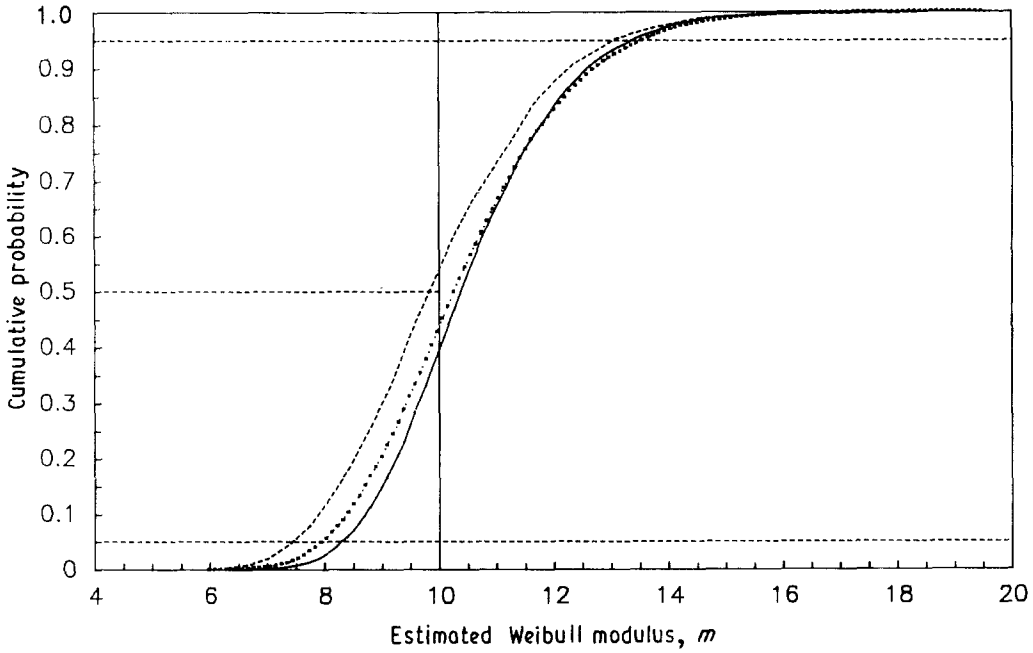


Figure 8 Cumulative distribution function of Weibull modulus, $N = 30$: (---) linear regression, $P_f = (n - 0.5)/N$; (···) moments method; (——) maximum likelihood method.

$N = 100$. Moreover, all three estimations have their peak values, i.e. the most commonly occurring estimator, equal to 10, the nominal value of m.
The PDFs of Fig. 7 can also be presented in the form of cumulative functions (Fig. 8). It turns out that

(i) for the method of linear regression with the definition $P_f = (n - 0.5)/N$, the median lies very close to the nominal value: there are as many overestimations among the estimators as underestimations;

(ii) the medians of the other two methods lie above $m = 10$, e.g. for the method of maximum likelihood one overestimates the parameter in about 60%, and not 50% of the cases.

From Fig. 8 one can also determine confidence intervals for the estimated parameter m, without having to make an approximation of m following a normal distribution

The 90% confidence interval for the three methods at a constant sample size of 30 were determined directly from Fig. 8 as follows:

Linear regression	$0.741m_{true} < m < 1.308m_{true}$
Moments method	$0.793m_{true} < m < 1.353m_{true}$
Maximum likelihood	$0.823m_{true} < m < 1.335m_{true}$

The smallest confidence interval is the one for the maximum likelihood method, which is herewith recommended by the authors as the most appropriate evaluation method.

4.1.1. The influence of m

The distributions represented in Figs 6 and 7 were all determined for an arbitrarily chosen value of $m = 10$. Previous authors (see e.g. [14]) reported that the value of m_{true} itself does not have an influence on the dispersion of the m distribution in all methods except the moments method.

Similar to Fig. 7, the distribution functions of m were simulated for all three methods, this time for three different values of m* (Fig. 9a–c). Fig. 9a–c show, however, that the dispersion of the parameter m is independent of the value assigned to m_{true} for all three evaluation methods, i.e. in order to study the estimator properties it is sufficient to do so for one value of m_{true}. The results are then valid for all values of m_{true}.

4.2. The estimator σ_0

It was already reported by previous authors that no significant variation occurs in determining the scale parameter σ_0. It can be expected that as for the parameter m an increasing sample size results in decreasing error in estimation of σ_0. This was reported by Schweiger *et al.* [12]. Hence the distributions of the scale factor estimators were merely considered for a constant samples size of 30. Fig. 10 shows the PDF, $f(\sigma_0)$, of 10000 estimators from the three evaluation methods.

All three methods seem to result in more or less equally dispersed symmetrical histograms. To compare the dispersion of the estimators for the two parameters m and σ_0, the PDFs of m from Fig. 7 and σ_0 from Fig. 10 by maximum likelihood evaluation are shown in Fig. 11†. Clearly, σ_0 may be determined with a higher degree of accuracy than m. Table I lists

* Repetitions of the simulated data were avoided as described in the first footnote.
† Note that the depicted PDFs were generated for the same x interval: 0.6 × true parameter as the lower bound, and 1.6 × true parameter as the upper bound. Only when this condition is fulfilled is it allowable to compare the relative frequencies (the y values). This is simply due to the fact that in order to make a comparison, the area under both curves has to be equal to unity on the same scale.

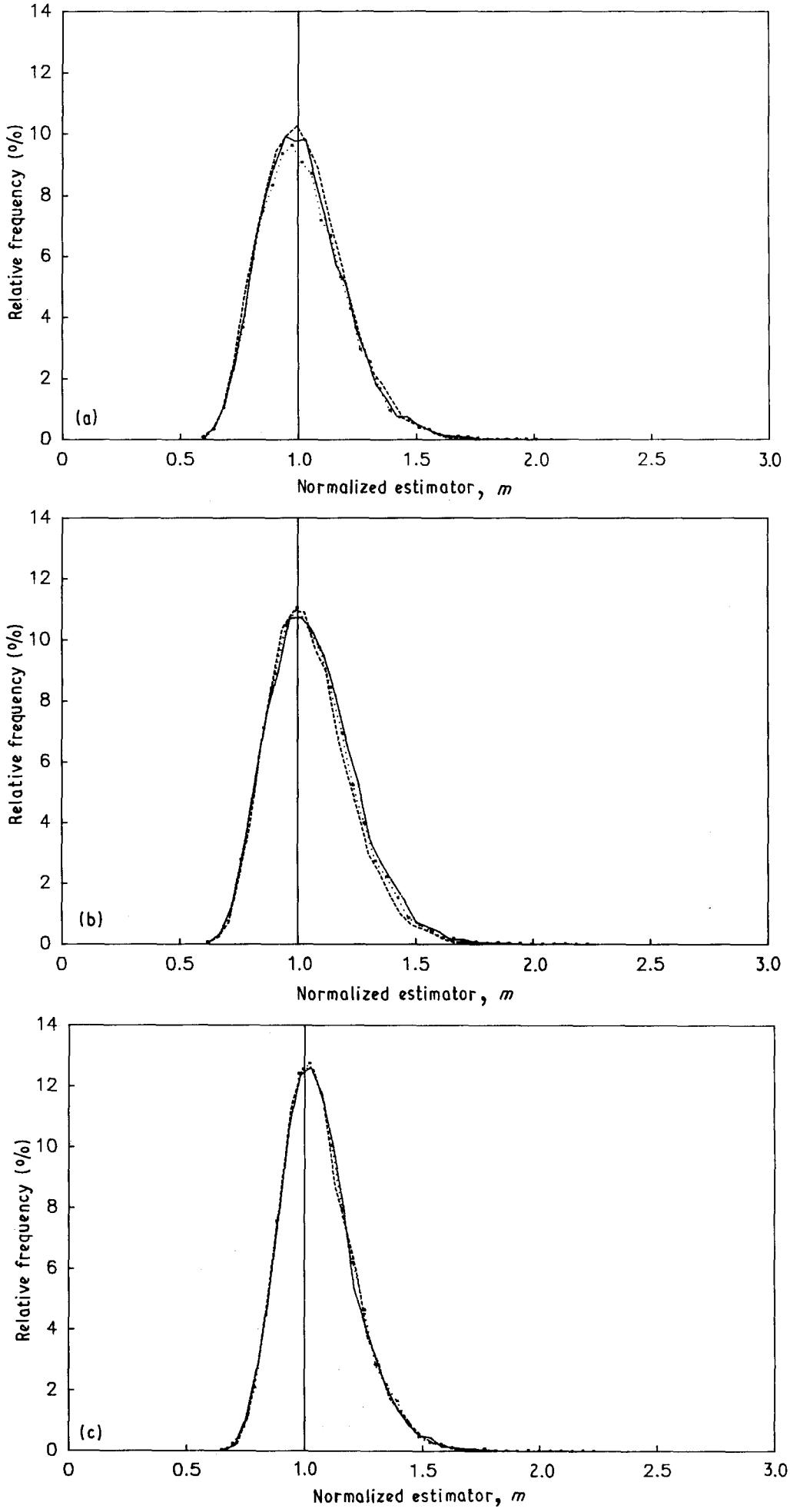


Figure 9 Influence of m_{true} on the probability density function, $N = 30$: (a) linear regression, (b) moments method, (c) maximum likelihood evaluation; (---) $m_{true} = 10$, (···) $m_{true} = 20$, (——) $m_{true} = 100$.

TABLE I Statistical properties of Weibull estimators for $N = 30$, $m_{true} = 10$, $\sigma_{0,true} = 1$

Estimation method	Average		Median		C_{var}	
	m	σ_0	m	σ_0	m	σ_0
Linear regression $P_f = (n - 0.5)/N$	10.209	1.0003	10.056	1.0007	0.1733	0.019 56
Moments method	10.484	0.9991	10.299	0.9995	0.1644	0.019 26
Maximum likelihood	10.594	0.9988	10.424	0.9990	0.1470	0.019 12

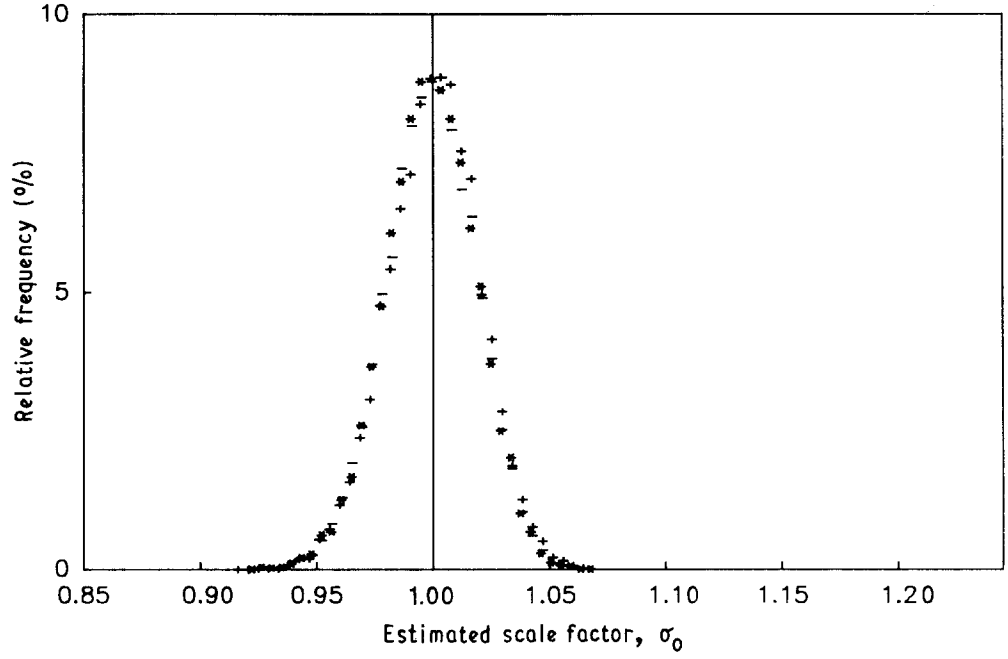


Figure 10 Probability density function of the scale factor σ_0, $N = 30$: (+) linear regression, $P_f = (n - 0.5)/N$; (−) moments method; (∗) maximum likelihood method.

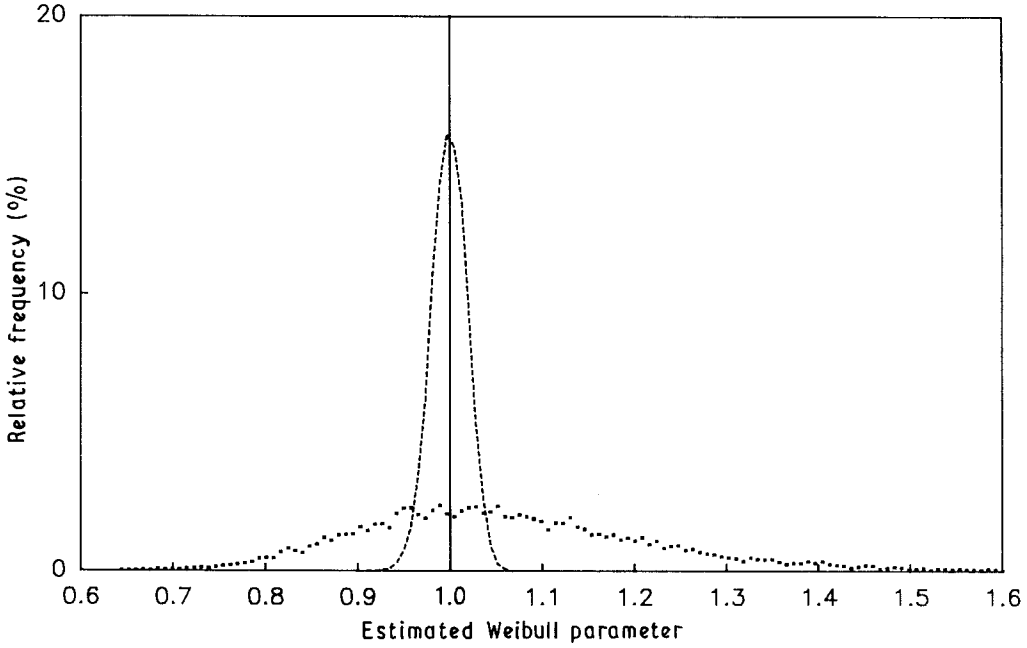


Figure 11 Probability density function of Weibull parameters estimated by the maximum likelihood method, $N = 30$ (10 000 estimators divided into 150 classes): (■) Weibull modulus, m; (---) scale factor, σ_0.

the average, the median, and the coefficient of variation C_{var} for the distributions of m and σ_0 at a constant sample size of 30.

The median value is smaller than the average value for the m distributions. This is to be expected from the asymmetry of the PDFs. For the σ_0-distributions, which also show a slight asymmetry, the opposite holds true: the estimator σ_0 tends rather to be underestimated than overestimated. As already mentioned, an underestimation of σ_0 leads to a steeper distribution.

As can be seen from Table I for both m and σ_0-distributions, the median value is closer to the true value of the parameter, i.e. all methods are less biased

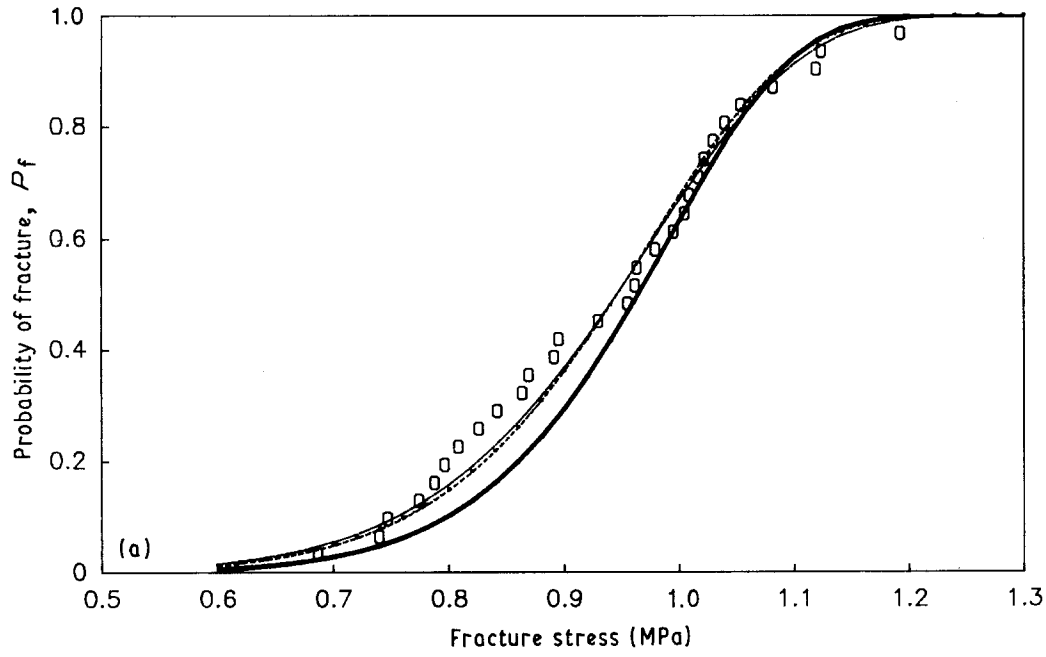
in their estimation than was assumed in previous work. As for the m distribution, the variance – the reproducibility – of the true parameter σ_0 is best for the maximum likelihood estimators.

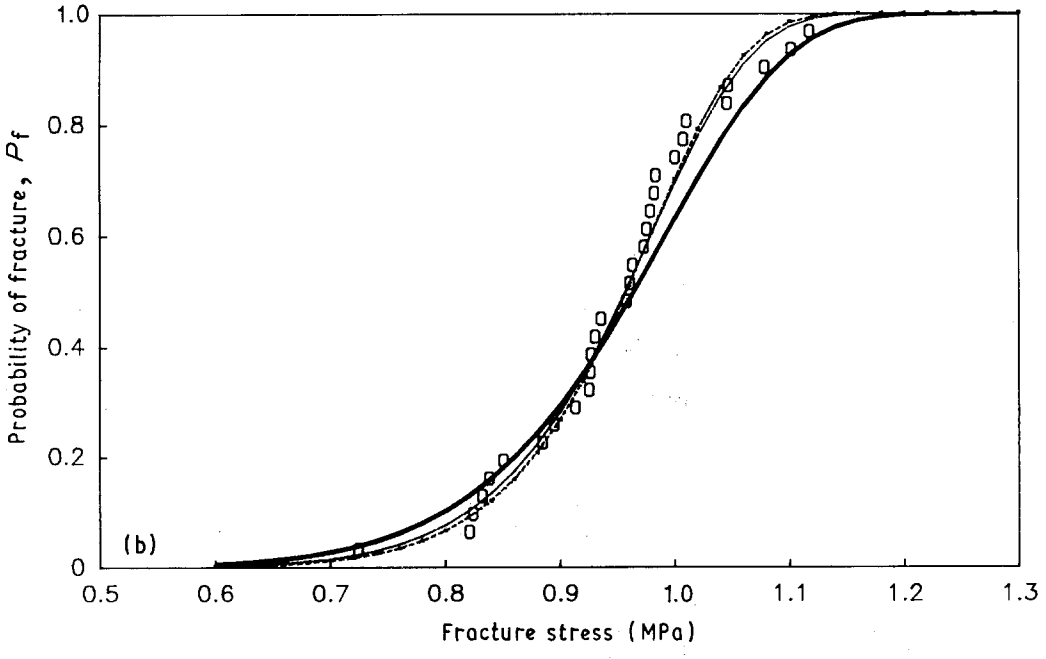
The parameter σ_0 is in the exponent's argument, $(\sigma/\sigma_0)^m$, in the Weibull equation. Hence its dispersion may still exert an important effect in the form of the Weibull cumulative function despite the small variance from its true value.

4.3. Examples

It is interesting to see the joint effect of the dispersion of both parameters in estimating the cumulative function. Fig. 12a–d show four (from 10000 simulations) selected cases of simulated experiments: the parent population (thick curve), the 30 simulated fracture stresses: $\{\sigma_1, \ldots, \sigma_{30}\}$ to each of which a probability of fracture P_f according to Equation 4a was assigned in drawing the diagram, and the Weibull distributions from the three estimation methods. Table II lists the parameters of all three estimated Weibull functions for each diagram.

The cases listed in Table II are arbitrarily chosen. All three methods seem to result in estimated Weibull curves very closely related to each other. How well a particular set of σ values represent the parent population determines the final shape and relevance of the estimated curve.





(*Figure 12a, b*)

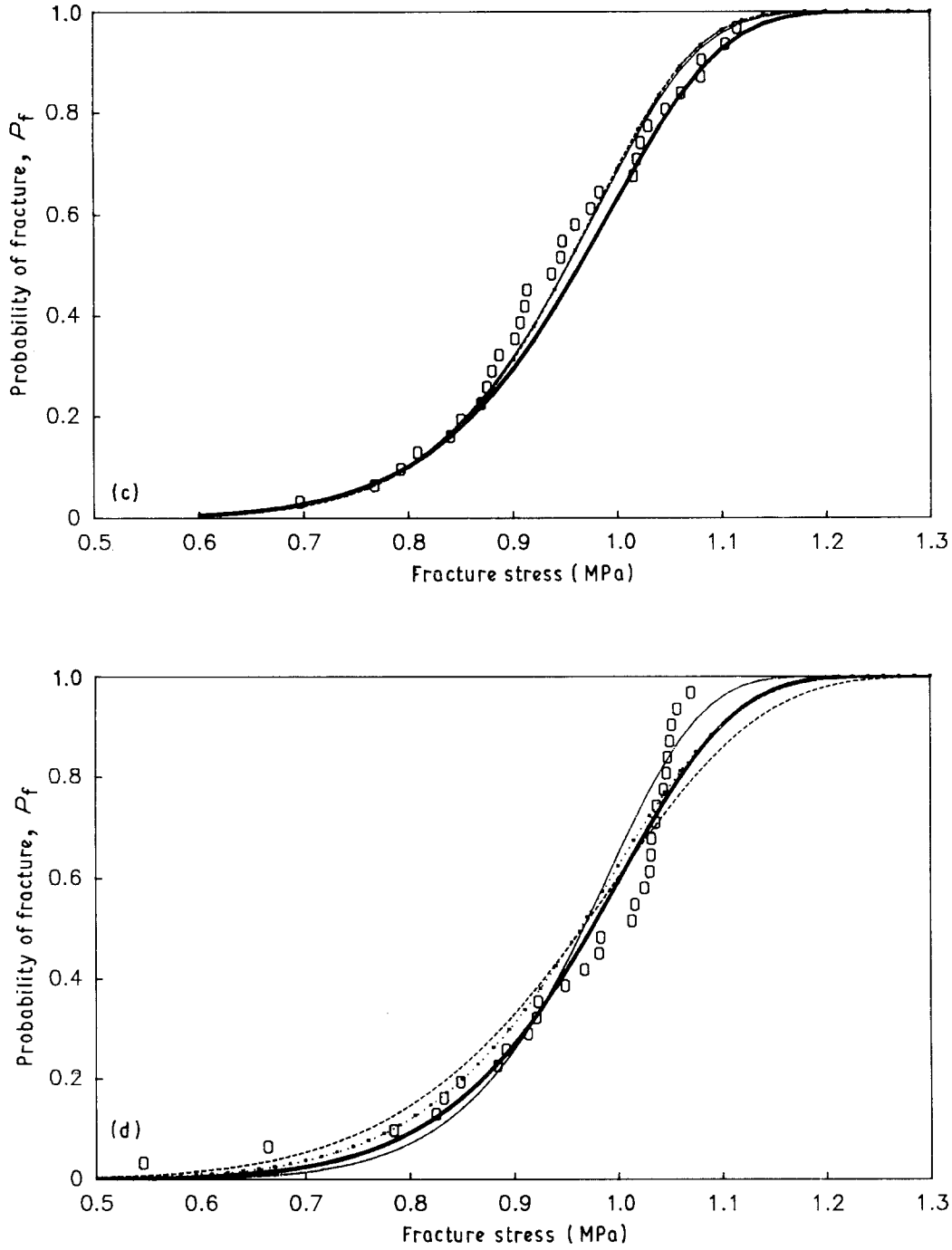


Figure 12(a–d) Estimated Weibull distributions for simulated experiments, $N = 30$. Four arbitrarily chosen cases (see Table II). (□) Simulated fracture stress, (——) parent population, (– – –) linear regression estimator, (···) moments estimator, (——) maximum likelihood estimator.

TABLE II Estimated Weibull parameters for four arbitrarily chosen cases

Estimation method	Fig. 12a		Fig. 12b		Fig. 12c		Fig. 12d	
	m	σ_0	m	σ_0	m	σ_0	m	σ_0
Linear regression $P_f = (n - 0.5)/N$	8.79	0.9861	12.84	0.9852	10.86	0.9849	7.89	1.0109
Moments method	8.70	0.9869	12.94	0.9852	10.84	0.9854	9.14	1.0026
Maximum likelihood	8.33	0.9880	12.10	0.9860	10.54	0.9862	11.86	0.9967

5. Concluding remarks

To obtain the Weibull function describing a set of experimental data best, i.e. the most correct estimation, it is important to characterize the statistical properties of the different estimators. Knowing the exact underlying Weibull distribution for a material, one is then able to utilize this material in the best way, and also study the effect of different production para-

meters or subsequent treatments on the fracture behaviour of the material through comparison of the Weibull distributions for the different cases.

Monte Carlo simulations were used to characterize the statistical properties of estimators resulting from the three evaluation methods: linear regression, moments method, and the method of maximum likelihood. Firstly, it was reaffirmed that the dispersion of the estimator for sample sizes smaller than 30 is much too high, i.e. for smaller sample sizes the uncertainty and the confidence interval in the obtained estimators are much too large. As a compromise between minimizing both the dispersion of the estimator on one hand, and the experimental effort on the other hand, the authors suggest a sample size of 30.

To characterize the statistical properties of the estimators, not only the average and the standard deviations were determined, but also histograms (density functions) of 10000 estimators were generated. It was shown that the distribution of the estimators takes an asymmetrical form. The median of the distribution was therefore used to compare the evaluation methods. For all estimators, the median lies closer to the true value of the parameter than the average, i.e. the estimators are less biased than assumed up to now.

The recommendation of other authors was also reaffirmed, that for the method of linear regression, the definition of probability of fracture most commonly used, $P_f = n/(N + 1)$, produces the least acceptable outcome. The definition $P_f = (n - 0.5)/N$ results in better estimators.

Furthermore, it was shown that the maximum likelihood estimator demonstrates the narrowest distribution for all sample sizes. Yet the probability that the maximum likelihood estimator over- or underestimates the true Weibull modulus is not 50:50 but about 60:40. This means that the maximum likelihood estimator results more often in an overestimation than underestimation. Yet since this estimator leads to the least dispersion, i.e. best reproducibility, for all sample sizes, it is recommended by the authors at this point.

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